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Minimum Degree and Density of Binary Sequences

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Abstract

For $d, k \in \mathbb{N}$ with $k \leq 2d$, let $g(d, k)$ denote the infimum density of binary sequences $(x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ which satisfy the minimum degree condition $\sum\limits_{j=1}^{d}(x_{i+j} + x_{i-j}) \geq k$ for all $i \in \mathbb{Z}$ with $x_i = 1$. We reduce the problem to determine $g(d, k)$ to a combinatorial problem related to the generalized k-girth of a graph G which is defined as the minimum order of an induced subgraph of G of minimum degree at least k. Extending results of Kézdy and Markert, and of Bermond and Peyrat, we present a minimum mean cycle formulation which allows to determine $g(d, k)$ for small values of d and k. For odd values of k with $d + 1 \leq k \leq 2d$, we conjecture $g(d, k) = \frac{k^2-1}{2(dk-1)}$ and show that this holds for $k \geq 2d - 3$.

Keywords: Minimum degree; density; binary sequence; girth; generalized girth; power of cycle

Proposed running head: "Degree and Density of Sequences"

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1 Introduction

Let $d \in \mathbb{N}$ be fixed. For a two-way infinite binary sequence

$$X = (x_i)_{i \in \mathbb{Z}} = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \{0,1\}^{\mathbb{Z}},$$

we define *the minimum degree* $\delta(X)$ *of* X as

$$\delta(X) = \min \left\{ \sum_{j=1}^{d} (x_{i+j} + x_{i-j}) \mid i \in \mathbb{Z}, x_i = 1 \right\}.$$

If $x_i = 0$ for all $i \in \mathbb{Z}$, then we write $X = 0$ and call X *trivial*.

For $k \in \mathbb{N}$ with $k \leq 2d$, we consider the infimum density $g(d,k)$ of non-trivial binary sequences subject to a minimum degree condition defined as

$$g(d,k) = \inf \left\{ \liminf_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} x_i \mid X = (x_i)_{i \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}, X \neq 0, \delta(X) \geq k \right\}.$$

Considering the binary sequence $(x_i)_{i \in \mathbb{Z}}$ with $x_i = 1$ if and only if $1 \leq i \leq k+1$, it follows that $g(d,k) = 0$ for $k \leq d$. While for such values of k, the calculation of $g(d,k)$ is trivial, for $k \geq d+1$, the calculation of $g(d,k)$ leads to an interesting combinatorial problem.

We prove as our first result that we can restrict ourselves to periodic sequences whose period is bounded in terms of d. Note that $g(d, 2d) = 1$ for all $d \in \mathbb{N}$.

Theorem 1 *Let $d, k \in \mathbb{N}$ with $d \geq 2$ and $d+1 \leq k \leq 2d$. There is a non-trivial periodic binary sequence $X = (x_i)_{i \in \mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and*

$$g(d,k) = \frac{1}{p} \sum_{j=1}^{p} x_j.$$

Proof: Let $0 < \epsilon < \frac{1}{3}$. Let $X = (x_i)_{i \in \mathbb{Z}}$ be a non-trivial binary sequence such that $\delta(X) \geq k$ and $\liminf_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} x_j \leq g(d,k) + \epsilon$. Since $x_i = 1$ for infinitely many $i \in \mathbb{Z}$, we have

$$\liminf_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} x_j \geq \frac{1}{2} \left(\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} x_j + \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} x_{-j} \right).$$

By symmetry, we may assume that $\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} x_j \leq g(d,k) + \epsilon$.

Note that $\delta(X) \geq k \geq d+1$ implies that X does not contain d consecutive 0-entries.

We call some $n \in \mathbb{N}$ *good* if

- $\frac{1}{n} \sum_{j=1}^{n} x_j \leq g(d,k) + 2\epsilon$ and

- $(x_{j_1}, x_{j_1+1}, \ldots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \ldots, x_{j_2+2d-1})$ for some $1 \leq j_1 \leq \lfloor \epsilon n \rfloor - 2d + 1$ and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

Claim *There are infinitely many good $n \in \mathbb{N}$.*

Proof of the claim: Let $n_1, n_2, \ldots, n_{2^{2d}} \in \mathbb{N}$ be such that $\frac{1}{n_i} \sum\limits_{j=1}^{n_i} x_j \leq g(d,k) + 2\epsilon$ for $1 \leq i \leq 2^{2d}$, $2d \leq \lfloor \epsilon n_1 \rfloor$, and $n_i \leq \lfloor \epsilon n_{i+1} \rfloor$ for $1 \leq i \leq 2^{2d} - 1$. Clearly, it suffices to prove that one of the n_i's is good. For contradiction, we assume that all n_i's are bad. Inductively, this implies that for $1 \leq i \leq 2^{2d}$, the sequence $(x_j)_{j \in \{1,2,\ldots,\lfloor \epsilon n_i \rfloor\}}$ contains i distinct subsequences of the form $(x_j, x_{j+1}, \ldots, x_{j+2d-1})$ with $1 \leq j \leq \lfloor \epsilon n_i \rfloor - 2d + 1$ which are different from all subsequences of the form $(x_j, x_{j+1}, \ldots, x_{j+2d-1})$ with $n_i - \lfloor \epsilon n_i \rfloor + 1 \leq j \leq n_i - 2d + 1$. Since there are exactly 2^{2d} distinct binary sequences of length d, this is impossible for $i = 2^{2d}$, which completes the proof of the claim. $\square$

Let $n \in \mathbb{N}$ be good. Let $(x_{j_1}, x_{j_1+1}, \ldots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \ldots, x_{j_2+2d-1})$ for $1 \leq j_1 \leq \lfloor \epsilon n \rfloor - 2d + 1$ and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

The non-trivial periodic binary sequence $X' = (x'_i)_{i \in \mathbb{Z}}$ with $x'_i = x_i$ for $j_1 + 2d \leq i \leq j_2 + 2d - 1$ of period $p' = j_2 - j_1$ satisfies $\delta(X') \geq k$ and

$$\frac{1}{p'} \sum_{j=1}^{p'} x'_j \leq \frac{1}{1 - 2\epsilon} \left(g(d,k) + 2\epsilon\right).$$

If $p' > 2d2^{2d}$, then the pigeonhole principle implies the existence of indices $1 \leq j_1, j_2 \leq p'$ with $(x'_{j_1}, x'_{j_1+1}, \ldots, x'_{j_1+2d-1}) = (x'_{j_2}, x'_{j_2+1}, \ldots, x'_{j_2+2d-1})$ and $j_1 + 2d \leq j_2 \leq j_1 + p' - 2d$. Let $X'' = (x''_i)_{i \in \mathbb{Z}}$ be the non-trivial p''-periodic binary sequence with $x''_i = x'_i$ for $j_1 + 2d \leq i \leq j_2 + 2d - 1$ with $p'' = j_2 - j_1$. Similarly, let $X''' = (x'''_i)_{i \in \mathbb{Z}}$ be the non-trivial p'''-periodic binary sequence with $x'''_i = x'_i$ for $j_2 + 2d \leq i \leq j_1 + p' + 2d - 1$ with $p''' = j_1 + p' - j_2$. Clearly, $p'', p''' < p'$, $\delta(X''), \delta(X''') \geq k$, and either $\frac{1}{p''} \sum\limits_{j=1}^{p''} x''_j \leq \frac{1}{1-2\epsilon}(g(d,k) + 2\epsilon)$ or $\frac{1}{p'''} \sum\limits_{j=1}^{p'''} x'''_j \leq \frac{1}{1-2\epsilon}(g(d,k) + 2\epsilon)$. This implies that for every $0 < \epsilon < \frac{1}{3}$, there is a non-trivial periodic binary sequence $X = (x_i)_{i \in \mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and $\frac{1}{p} \sum\limits_{j=1}^{p} x_j \leq \frac{1}{1-2\epsilon}(g(d,k) + 2\epsilon)$. Since for every such sequence X, the quantity $\frac{1}{p} \sum\limits_{j=1}^{p} x_j$ is a rational number whose denominator is bounded by $d2^{2d+1}$, the desired result follows. $\square$

For the further investigations, it is more convenient to consider a cyclic binary sequence

$$X = (x_0, x_1, \ldots, x_{p-1}) = x_0 x_1 \ldots x_{p-1}$$

of length p instead of a periodic binary sequence $(x_i)_{i \in \mathbb{Z}}$ with period p. As usual, we will consider indices modulo the length p. We say that an entry x_i of X *sees* another entry x_j of X if the cyclic distance of x_i and x_j is at least 1 and at most d. To avoid double-counting, we define *the minimum degree* $\delta(X)$ *of* X as the minimum number of distinct 1-entries of X seen by a 1-entry of X. Furthermore, we define *the density* $\mu(X)$ *of* X as

$$\mu(X) = \frac{1}{p}\sum_{j=1}^{p} x_j.$$

With these notions, Theorem 1 implies that $g(d,k)$ equals the minimum density of a non-trivial cyclic binary sequence X of length at most $d2^{2d+1}$ and minimum degree $\delta(X) \geq k$.

Our original motivation to study $g(d,k)$ comes from graph theory: For a finite, simple and undirected graph $G = (V,E)$ and $k \in \mathbb{N}$, the *k-girth* $g_k(G)$ *of* G is the minimum order of an induced subgraph of G of minimum degree at least k. The notion of k-girth was proposed and studied by Erdős et al. [3–5] and Bollobás and Brightwell [2]. It generalizes the usual girth, the length of a shortest cycle, which coincides with the 2-girth.

Kézdy and Markert studied bounds on this generalized girth [7, 8]. They conjectured that the d-th power of the cycle of length $n \geq 2d+1$, denoted by C_n^d, is the $2d$-regular graph with largest $(d+1)$-girth [8] (see also Chapter 5 of [7]). During the 1988 SIAM conference, Kézdy [6] posed the problem to determine the exact value of the $(d+1)$-girth of C_n^d. For odd values of d, this problem was solved by Bermond and Peyrat [1] who proved that for $d+1 \leq k \leq 2d$, the k-girth of C_n^d satisfies

$$\frac{g_k\left(C_n^d\right)}{n} \geq \frac{k}{2d}. \tag{1}$$

The bound (1) is best-possible whenever k is even in view of the induced subgraph of C_n^d where n is a multiple of d and which alternately contains $\frac{k}{2}$ consecutive vertices of C_n^d and does not contain the next $d - \frac{k}{2}$ consecutive vertices of C_n^d. For odd values of k, Bermond and Peyrat mentioned results for some small values of d and k, and proved the best-possible estimate $\frac{g_{2d-1}(C_n^d)}{n} \geq \frac{2d}{2d+1}$.

An induced subgraph G of C_n^d can be conveniently identified with a cyclic binary sequence $X = (x_0, x_1, \ldots, x_{n-1})$ of length n where 1-entries correspond to vertices of C_n^d which belong to G and 0-entries correspond to vertices of C_n^d which do not belong to G. This correspondence implies that $g(d,k)$ equals the minimum k-girth of the d-th power of cycles, i.e.

$$g(d,k) = \min\left\{\frac{g_k(C_n^d)}{n} \mid n \geq 3\right\}.$$

The above-mentioned results of Bermond and Peyrat imply that $g(d,k) = \frac{k}{2d}$ for $d+1 \leq k \leq 2d$ with even k and that $g(d, 2d-1) = \frac{2d}{2d+1}$. Kézdy and Markert determined $g(4,5) = \frac{12}{19}$ and $g(6,7) = \frac{24}{41}$ with the help of a computer. Bermond and Peyrat [1] claimed that $g(5,7) = \frac{5}{7}$ which is not correct (see Section 2). Furthermore, they conjectured that

$$g(d,k) = \frac{d(2d+3-k)}{2(d^2 - (k-d-2)d - (k-d))}$$

for $d+1 \leq k \leq 2d$ with k odd. Since this expression is less than $\frac{k}{2d}$ if and only if $\left|k - \frac{3d}{2}\right| < \frac{d}{2}\sqrt{1 - \frac{4}{d+1}}$, this conjecture is obviously not correct in view of (1).

Our results are as follows. In Section 2, we explain how for fixed values of d and k, the problem to determine $g(d,k)$ can be reduced to a minimum mean cycle problem on a suitably defined directed graph with arc costs. This allows to determine $g(d,k)$ and also the structure of optimal subgraphs of C_n^d for many small values of d and k and motivates a corresponding conjecture explained in Section 3. Moreover, in Section 4, we prove as our main result that our conjecture is true for $k = 2d-3$, i.e. we determine $g(d, 2d-3)$.

2 Minimum Mean Cycle Formulation

Given a directed graph $D = (V, A)$ and a cost function $c : A \to \mathbb{R}$, a *minimum mean cycle* is a directed cycle

$$C : v_1 v_2 \dots v_n v_1$$

in D for which

$$\overline{c}(A(C)) = \frac{1}{n} \sum_{a \in A(C)} c(a)$$

is minimum. Karp [9] observed that a minimum mean cycle can be found efficiently using shortest path methods.

For $d \in \mathbb{N}$ and $d+1 \leq k \leq 2d$, let $D = (V, A)$ be the directed graph whose vertex set V consists of all binary sequences

$$(x_{-d}, \dots, x_{-1}, x_0, x_1, \dots, x_d)$$

of length $2d+1$ with $x_0 = 1$ and $\sum\limits_{i=1}^{d}(x_i + x_{-i}) \geq k$ and which contains a directed arc (x,y) of cost $c((x,y)) = -i^*$ from a vertex $x = (x_{-d}, \dots, x_d)$ to a vertex $y = (y_{-d}, \dots, y_d)$ exactly if

$$(x_{i^*-d}, \dots, x_0, \dots, x_{i^*}, \dots, x_d) = (y_{-d}, \dots, y_{-i^*}, \dots, y_0, \dots, y_{d-i^*})$$

for $i^* = \min\{i \mid 1 \leq i \leq d, x_i = 1\}$. Note that i^* is well-defined and that the last condition implies that x and y can be suitably overlayed, i.e. there is a binary sequence z of length $2d+1+i^*$ such that x corresponds to the first $2d+1$ entries of z and y corresponds to the last $2d+1$ entries of z. See Figure 1 for an illustration.

Theorem 2 *If D and c are as above and C is a minimum mean cycle of D, then*

$$g(d,k) = -\frac{1}{\overline{c}(A(C))}.$$

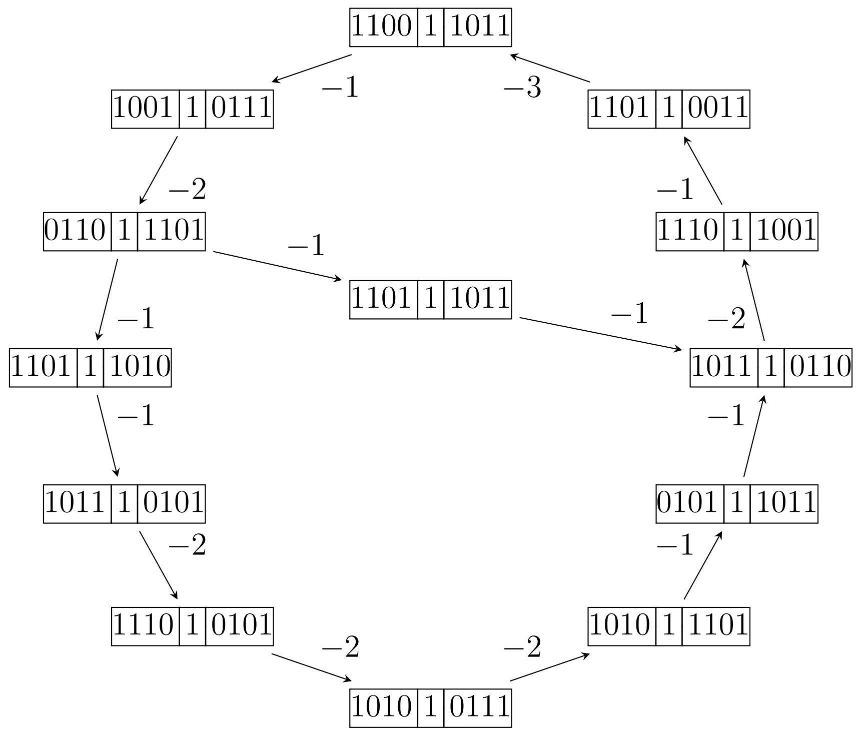

Figure 1: Induced subgraph of the directed graph D for $d = 4$ and $k = 5$.

Proof: Clearly, for every directed cycle $C : v_1v_2 \dots v_nv_1$ in D, suitably overlaying the sequences $v_1, v_2, \dots, v_n$ — as x and y above — results in a cyclic binary sequence X with $\delta(X) \geq k$. Since the number of 1-entries of X equals n and the length of X equals $-\sum\limits_{a \in A(C)} c(a)$, we obtain $\mu(X) = -\frac{1}{\overline{c}(A(C))}$.

Conversely, if X is a cyclic binary sequence with $\delta(X) \geq k$, the sequences of length $2d+1$ centered at the consecutive 1-entries of X define a directed closed walk W in D. By Euler's theorem, W contains a directed cycle C with $\overline{c}(A(C)) \leq \overline{c}(A(W))$. Since the length of W equals the number of 1-entries of X and the length of X is $-\sum\limits_{a \in A(C)} c(a)$, we obtain $\overline{c}(A(C)) \leq \overline{c}(A(W)) = -\frac{1}{\mu(X)}$.

These two observations clearly imply the desired result. $\square$

Table 1 summarizes some explicit values of $g(d, k)$ obtained by this approach together with realizing cyclic binary sequences. In fact, we determined optimal sequences for all values of d and $d + 1 \leq k \leq 2d$ with $d \leq 13$, $k \geq 2d - 7$, and k odd. For $(d, k) = (5, 7)$ for example, we obtained $g(5, 7) = \frac{24}{34}$, and a realizing cyclic binary sequence is

$$(1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0),$$

which we write shortly as $1^3 0 1^4 0 1 0 1^2 0 1^4 0 1^2 0 1 0 1^4 0 1^3 0^2$.

$2d$-k	(d,k)	$g(d,k)$	Optimal cyclic sequences, candidates for $\mathbf{u}$ highlighted
3	(4, 5)	12/19	$\mathbf{1^201}\ 1^201\ 0\ \mathbf{101^2}\ 101^2\ 0^2$
	(5, 7)	24/34	$\mathbf{1^301}\ 1^301\ 0\ \mathbf{1^201^2}\ 1^201^2\ 0\ \mathbf{101^3}\ 101^3\ 0^2$
	(6, 9)	40/53	$\mathbf{1^401}\ 1^401\ 0\ \mathbf{1^301^2}\ 1^301^2\ 0\ \mathbf{1^201^3}\ 1^201^3\ 0\ \mathbf{101^4}\ 101^4\ 0^2$
	(7, 11)	60/76	$\mathbf{1^501}\ 1^501\ 0\ \mathbf{1^401^2}\ 1^401^2\ 0\ \mathbf{1^301^3}\ 1^301^3\ 0\ \mathbf{1^201^4}\ 1^201^4\ 0\ \mathbf{101^5}\ 101^5\ 0^2$
5	(6, 7)	24/41	$\mathbf{10^21^3}\ 10^21^3\ 0^3\ \mathbf{1^30^21}\ 1^30^21\ 0\ \mathbf{1^20^21^2}\ 1^20^21^2\ 0$
			$\mathbf{101^201}\ 101^201\ 0^2\ \mathbf{1^20101}\ 1^20101\ 0\ \mathbf{10101^2}\ 10101^2\ 0^2$
	(7, 9)	40/62	$\mathbf{1^40^21}\ 1^40^21\ 0\ \mathbf{1^30^21^2}\ 1^30^21^2\ 0\ \mathbf{1^20^21^3}\ 1^20^21^3\ 0\ \mathbf{10^21^4}\ 10^21^4\ 0^3$
			$\mathbf{1^30101}\ 1^30101\ 0\ \mathbf{1^20101^2} 1^20101^2\ 0\ \mathbf{10101^3}\ 10101^3\ 0^2\ \mathbf{101^301}\ 101^301\ 0^2$
		20/31	$\mathbf{101^201^2}\ 101^201^2\ 0^2\ \mathbf{1^201^201}\ 1^201^201\ 0$
7	(8, 9)	40/71	$\mathbf{1^40^31}\ 1^40^31\ 0\ \mathbf{1^30^31^2}\ 1^30^31^2\ 0\ \mathbf{1^20^31^3}\ 1^20^31^3\ 0\ \mathbf{10^31^4}\ 10^31^4\ 0^4$
			$\mathbf{1^20^21^201}\ 1^20^21^201\ 0\ \mathbf{10^21^201^2}\ 10^21^201^2\ 0^3\ \mathbf{1^201^20^21}\ 1^201^20^21\ 0\ldots$
			$\mathbf{101^30^21}\ 101^30^21\ 0^2\ \mathbf{1^30^2101}\ 1^30^2101\ 0\ \mathbf{1^20^2101^2}\ 1^20^2101^2\ 0\ldots$
			$\mathbf{101^20101}\ 101^20101\ 0^2\ \mathbf{1^2010101}\ 1^2010101\ 0\ \mathbf{1010101^2}\ 1010101^2\ 0^2\ldots$

Table 1

3 A Conjecture for $g(d,k)$

We have observed that all optimal sequences that we have computed can be obtained by applying a uniform construction rule.

Let U be the set of finite binary sequences starting and ending with a 1. For $\mathbf{u} \in U$ with $\mathbf{u} = 10^a\mathbf{v}$ for some $\mathbf{v} \in U$, the *shift operation* s applied to $\mathbf{u}$ results in $s(\mathbf{u}) = \mathbf{v}0^a1$, i.e. it removes all entries of $\mathbf{u}$ before the second 1 and appends them at the end in reverse order. For $\mathbf{u} = 11101$, for example, we obtain

$$s(\mathbf{u}) = 11011,\ \ s^2(\mathbf{u}) = s(s(\mathbf{u})) = 10111, \ \text{ and } \ s^3(\mathbf{u}) = 11101 = \mathbf{u}.$$

For $d,k \in \mathbb{N}$ with $d+1 \le k \le 2d$ and k odd, let U_k^d be the set of those sequences in U with length d and exactly $l = \frac{k+1}{2}$ many 1-entries.

Note that for $\mathbf{u} \in U_k^d$, we have $s^{l-1}(\mathbf{u}) = \mathbf{u}$.

The *shifted sequence* for $\mathbf{u}$ is the concatenation

$$X(\mathbf{u}) = \mathbf{u}\mathbf{u}0^{a_1+1}s(\mathbf{u})s(\mathbf{u})0^{a_2+1}\ldots 0^{a_{l-2}+1}s^{l-2}(\mathbf{u})s^{l-2}(\mathbf{u})0^{a_{l-1}+1},$$

where a_i is the number of 0s between the i-th and $(i+1)$-st 1-entry of $\mathbf{u}$, i.e. $\mathbf{u} = 10^{a_1}10^{a_2}1\ldots 10^{a_{l-1}}1$. For $\mathbf{u} = 11011 \in U_7^5$, we have

$$X(\mathbf{u}) = 11011\,11011\,0\,10111\,10111\,00\,11101\,11101\,0$$

which is a cyclic shift of the sequence for $(5,7)$ in Table 1.

A subsequence of consecutive entries of a cyclic binary sequence is called an *interval*.

Lemma 3 *Let $d,k \in \mathbb{N}$ be such that $d+1 \le k \le 2d$ and k is odd. Let $\mathbf{u} \in U_k^d$.*

(i) $X(\mathbf{u})$ has length $dk-1$,

(ii) $\mu(X(\mathbf{u})) = \frac{k^2-1}{2(dk-1)}$,

(iii) $\delta(X(\mathbf{u})) = k$, *and*

(iv) $g(d,k) \leq \mu(X(\mathbf{u}))$.

Proof: Let $\mathbf{u} = 10^{a_1}10^{a_2}1\ldots 10^{a_{l-1}}1$. The length of $X(\mathbf{u})$ equals

$$(l-1)2d + \sum_{i=1}^{l-1}(a_i+1) = (k-1)d + (d-1) = dk-1.$$

Furthermore, $X(\mathbf{u})$ contains $(l-1)2l = \frac{k^2-1}{2}$ many 1-entries. This implies (i) and (ii).

Note that the shifted sequences for $\mathbf{u}$ and for $s(\mathbf{u})$ are cyclic translates of each other. Furthermore, note that the reverse of a shifted sequence is also the cyclic translate of a shifted sequence. Therefore, in order to prove (iii), it suffices to consider the 1-entries within the first copy of $s(\mathbf{u})$ in $X(\mathbf{u})$.

By definition, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the first 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals (the central entry is highlighted)

$$10^{a_2}1\ldots 10^{a_{l-2}}10^{a_{l-1}}10^{a_1+1}\mathbf{1}0^{a_2}10^{a_3}1\ldots 10^{a_{l-1}}10^{a_1}11.$$

Hence this 1-entry sees $(l-1)$ 1-entries to the left and l 1-entries to the right, i.e. it sees $2l-1=k$ 1-entries.

For $2 \leq i \leq l-2$, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the i-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals

$$10^{a_{i+1}}1\ldots 10^{a_{l-1}}10^{a_1+1}10^{a_2}1\ldots 10^{a_i}\mathbf{1}0^{a_{i+1}}1\ldots 10^{a_{l-1}}10^{a_1}110^{a_2}1\ldots 10^{a_i}1.$$

Again this 1-entry sees $2l-1=k$ 1-entries.

The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the $(l-1)$-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $10^{a_1+1}10^{a_2}1\ldots 10^{a_{l-1}}\mathbf{1}0^{a_1}110^{a_2}1\ldots 10^{a_{l-1}}1$. Again this 1-entry sees $2l-1=k$ 1-entries.

The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the l-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $010^{a_2}1\ldots 10^{a_{l-1}}10^{a_1}\mathbf{1}s(\mathbf{u})$. Again this 1-entry sees $2l-1=k$ 1-entries.

(iv) follows immediately from (ii) and (iii). □

We pose the following conjecture.

Conjecture 4 *If $d \in \mathbb{N}$ and $d+1 \leq k \leq 2d$ are such that k is odd, then*

$$g(d,k) = \frac{k^2-1}{2(dk-1)}.$$

Furthermore, a cyclic binary sequence X with $\delta(X) \geq k$ has density $g(d,k)$ if and only if X is the concatenation of copies of a shifted sequence $X(\mathbf{u})$ for some $\mathbf{u} \in U_k^d$.

The case $k = 2d - 1$ of Conjecture 4 follows from the results and arguments in [1]. In this case U^d_{2d-1} contains only the element $\mathbf{u} = 1^d$ and $X(\mathbf{u}) = 1^{2d}01^{2d}0 \dots 1^{2d}0$.

Since we will prove Conjecture 4 for $k = 2d - 3$, it is useful to consider the structure of $X(\mathbf{u})$ for $\mathbf{u} \in U^d_{2d-3}$. In this case, $\mathbf{u}$ is a sequence of length d containing $(d - 1)$ 1-entries. If $\mathbf{u}^* = 10^{a_1}10^{a_2}1 \dots 10^{a_{l-1}}1$ with $a_1 = \dots = a_{l-2} = 0$ and $a_{l-1} = 1$, then $\mathbf{u}^* = 1^{d-2}01$ and

$$\begin{aligned} X(\mathbf{u}^*) &= 1^{d-2}01\,1^{d-2}01\,0\,1^{d-3}01^2\,1^{d-3}01^2\,0\,\dots 101^{d-2}\,101^{d-2}\,0^2 \\ &= 1^{d-2}01^{d-1}0101^{d-3}01^{d-1}01^201^{d-4}01^{d-1}01^30 \dots 101^{d-1}01^{d-2}0^2 \end{aligned}$$

Since for every $\mathbf{u} \in U^d_{2d-3}$, there is some i with $s^i(\mathbf{u}^*) = \mathbf{u}$, every shifted sequence $X(\mathbf{u})$ for $\mathbf{u} \in U^d_{2d-3}$ arises from $X(\mathbf{u}^*)$ by a cyclic shift. In this sense, the conjectured extremal sequences are unique.

4 The Value of $g(d, 2d - 3)$

Throughout this section let $d \geq 4$ and let $\mathcal{X}$ be the set of cyclic binary sequences X with $\delta(X) \geq 2d - 3$. This section is devoted to the proof of Conjecture 4 for $k = 2d - 3$, i.e. we will prove the following result.

Theorem 5 *Every $X \in \mathcal{X}$ satisfies $\mu(X) \geq \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Equality holds if and only if X is the concatenation of shifted sequences $X(\mathbf{u}^*)$ with $\mathbf{u}^* = 1^{d-2}01$.*

Before proving Theorem 5, we investigate structural properties of sequences in $\mathcal{X}$. Let

$$X = (x_0, x_1, \dots, x_{n-1}) = x_0x_1 \dots x_{n-1} \in \mathcal{X} \text{ with } n \geq 2d + 1.$$

Recall that an entry x_i of X sees another entry x_j of X, if x_j is in one of the intervals $x_{i-d}x_{i-d+1} \dots x_{i-1}$ or $x_{i+1}x_{i+2} \dots x_{i+d}$. We call x_i *regular* if it sees exactly $(2d-3)$ 1-entries and hence exactly three 0-entries. We first show that all irregular entries see more than $(2d - 3)$ 1-entries and describe the local structure around regular 0-entries.

Lemma 6

(i) All entries of X see at most three 0-entries.

(ii) For every regular 0-entry x_i, either $x_{i+1} = x_{i+d} = 0$, or $x_{i-1} = x_{i-d} = 0$, or $x_{i-d} = x_{i+d} = 0$.

Proof: (i): By assumption, all 1-entries of X see at most three 0-entries. For contradiction, we assume that some 0-entry of X sees more than three 0-entries. This implies that X has an interval $X' = 10^a1$ such that some 0-entry of X' sees at least four 0-entries. Since $d \geq 4$ and each of the two 1-entries of X' see at most three 0-entries, we obtain $a \leq 3$. Moreover, the two 1-entries of X' together see at most $(6 - a)$ distinct 0-entries. If $a \geq 2$, then every 0-entry of X' sees at most three 0-entries, a contradiction. Hence $a = 1$. If x_i

is the 0-entry in X', then each 1-entry of X' sees all but one entry seen by x_i. Thus it sees at least three 0-entries seen by x_i and the 0-entry x_i which is the final contradiction.

(ii): Again, the interval X' of the form 10^a1 of X containing the regular 0-entry x_i satisfies $a \leq 3$. If $a = 3$, then one of the two 1-entries of X' sees x_i and all three 0-entries seen by x_i which is a contradiction. If $a = 2$, then, by symmetry, we may assume that x_i is the first 0-entry of X'. Since the 1-entry x_{i-1} does not see one of the 0-entries seen by x_i, we have $x_{i+1} = x_{i+d} = 0$. Finally, if $a = 1$, then each of the 1-entries x_{i-1} and x_{i+1} does not see one of the 0-entries seen by x_i which implies $x_{i+d} = x_{i-d} = 0$ and completes the proof of (ii). $\Box$

Let n_1 denote the number of 1-entries of X. Moreover, let n^+ denote the number of irregular entries of X.

We can relate the density of X to the number of irregular entries of X.

Lemma 7

$$\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2dn}.$$

Proof: By Lemma 6 (i), double-counting the pairs (x_i, x_j) where $x_i = 1$ and x_i sees x_j yields $(2d-3)(n-n^+) + (2d-2)n^+ \leq 2dn_1$ which implies $\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2d}$. $\Box$

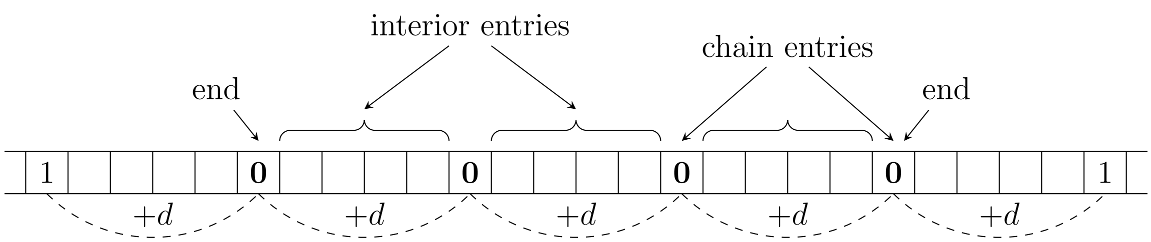

Figure 2: A chain of length 4 for $d = 5$.

A *chain* of X is a maximal subsequence

$$C = (x_i, x_{i+d}, \ldots, x_{i+kd})$$

of distinct 0-entries of X such that $k \geq 1$. A chain may be *cyclic* in which case $i \equiv i+(k+1)d \pmod{n}$. Otherwise C has two distinct *ends* x_i and x_{i+kd} where $x_{i-d} = 1 = x_{i+(k+1)d}$. Associated with the chain C are the *interior* entries of C, which are those entries that belong to one of the intervals $x_{i+jd+1}x_{i+jd+2}\ldots x_{i+jd+d-1}$, $0 \leq j \leq k-1$, between consecutive chain entries x_{i+jd} and $x_{i+(j+1)d}$ of C. We say that two chains *overlap*, if a chain entry of one chain is an interior entry of the second chain. Clearly, in this case, also a chain entry of the second chain is an interior entry of the first chain. Note that a chain may overlap itself.

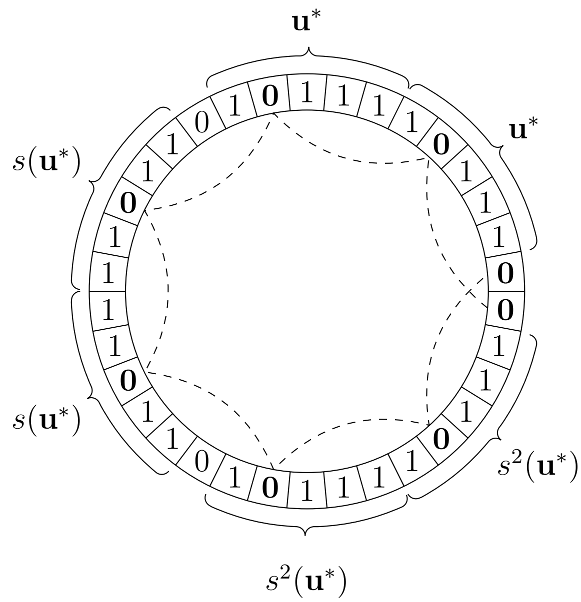

Figure 3: The example $X'(\mathbf{u}^*)$ for $d = 5$, i.e. with $\mathbf{u}^* = 11101$.

For example, the sequence $X'(\mathbf{u}^*) = x'_0 x'_1 \ldots x'_{n-1}$ which arises from the shifted sequence $X(\mathbf{u}^*)$ for $\mathbf{u}^* = 1^{d-2}01$ by moving the final 0-entry to the beginning

$$\begin{aligned} X'(\mathbf{u}^*) &= x'_0 x'_1 \ldots x'_{n-1} \\ &= \mathbf{0}1^{d-2}\mathbf{0}1^{d-1}\mathbf{0}101^{d-3}\mathbf{0}1^{d-1}\mathbf{0}1^2 01^{d-4}\mathbf{0}1^{d-1}\mathbf{0}1^3 0 \ldots 1\mathbf{0}1^{d-1}\mathbf{0}1^{d-2}\mathbf{0} \end{aligned} \tag{2}$$

has the single chain $C = (x'_{n-1}, x'_{d-1}, x'_{2d-1}, \ldots, x'_{n-d}, x'_0)$ whose ends x'_{n-1} and x'_0 are both interior entries as well as chain entries of C. See Figures 2 and 3 for an illustration.

We will show that chains may overlap only in their respective ends. More precisely, in Lemma 8 (ii) below we show that if x_i is a chain entry of C which is an interior entry of chain C' and x_{i-d} is another chain entry of C, then x_i is an end of C and x_{i-1} is an end of $C' = (x_{i-1}, x_{i+d-1}, \ldots)$. If this occurs, we call the interval $x_{i-1}x_i = 0^2$ a pair of *overlapping chain ends.*

Lemma 8

(i) Every regular 0*-entry of* X *belongs to some chain of* X*.*

(ii) If a chain entry of C *is an interior entry of the (not necessarily distinct) chain* C'*, then it belongs to a pair of overlapping chain ends.*

(iii) Let $x_{i-1}x_i$ *be a pair of overlapping chain ends. The intervals of length* $2d$ *ending and starting in* $x_{i-1}x_i = 0^2$ *have the form* $1^{d-1}01^{d-2}0^2$ *and* $0^2 1^{d-2}01^{d-1}$*, respectively.*

(iv) An end of a chain is regular in X *if and only if it belongs to a pair of overlapping chain ends.*

Proof: (i): This follows immediately from Lemma 6 (ii).

(ii): Let x_i be a chain entry of C which is an interior entry of C'. Then there must be chain entries x_j, x_{j+d} with $i-d<j<i$ of C'. By symmetry, we may assume that x_{i-d} is another chain entry of C. If $j<i-1$, then x_{i-1} sees at least four 0-entries, a contradiction. So $j=i-1$. Moreover, $x_{j-d}=1=x_{i+d}$, otherwise x_{i-2} or x_{i+1} sees four 0-entries. So x_i is an end of C and x_{i-1} is an end of C'.

(iii): Since both x_{i-1} and x_i already see three of the four 0-entries $x_{i-d}, x_{i-1}, x_i, x_{i+d-1}$, we obtain that $x_{i-d+1}=1=x_{i+d-2}$. Since each of these two entries sees three of the four 0-entries, too, all other entries seen by them must be 1, and the two intervals of X ending and starting in $x_{i-1}x_i$ have the required form.

(iv): It follows from (iii) that overlapping ends of chains are regular. Conversely, we assume that x_i is an end of a chain which is not an interior entry of any chain. By symmetry, we may assume that $x_{i-d}=0$ and $x_{i+d}=1$. If x_i is regular, then $x_{i-1}=0$, otherwise x_{i-1} sees x_i and all the three 0-entries seen by x_i, a contradiction to Lemma 6 (i). But since x_{i-1} does not belong to a chain, it must be irregular by (i) and thus x_{i-1} sees only the two 0-entries x_i and x_{i-d}. So x_i must be irregular as well. □

Lemma 9 *Let $I=x_{j-d}x_{j-d+1}\dots x_{j+d}$ be an interval of $2d+1$ entries of X.*

(i) If I contains no irregular entry, then I contains a regular end of a chain.

(ii) If I does not contain a regular chain end but contains an irregular chain end, then it contains at least two irregular entries.

Proof: (i): Since the center x_j of I is regular, it sees exactly three 0-entries, all of which are regular. By the length of I, only two of them can belong to the same chain. So, by Lemma 8 (iv), the third must be a regular chain end belonging to a pair of overlapping chain ends.

(ii): For contradiction, we assume that I contains exactly one irregular entry, an irregular chain end. If the center x_j is not the irregular chain end itself, then it is regular. So it sees two further 0-entries apart from the irregular chain end. Since these are regular, they all belong to chains. Hence, by Lemma 8 (ii), one of them is a regular chain end, a contradiction. So let x_j be the irregular chain end. We may assume that $x_{j-d}=0$. If x_j sees another 0-entry apart from x_{j-d}, then, by Lemma 8 (i) and (iv), this 0-entry is irregular. Otherwise, x_{j+1} is irregular, a contradiction. □

Lemma 10 *If X has a single chain whose ends overlap, then X has at least $d-3$ irregular entries.*

Proof: Let $(x_0, x_d, x_{2d}, \dots, x_{n-d+1}, x_1)$ be the chain and let $2\le r\le d-2$. We prove that there is some irregular entry x_j with $2\le j\le n-2$ and $j\equiv r \bmod d$.

If an entry at such a position satisfies $x_j = 0$, then, by Lemma 8 (i) and (ii), x_j is irregular. Hence, we may assume that $x_j = 1$ for all $2 \leq j \leq n-2$ with $j \equiv r \bmod d$. We choose a largest $s < r$ such that X has an entry $x_k = 0$ with $k \equiv s \bmod d$. Note that $x_1 = 0$ implies that s is well-defined and that $1 \leq s < r$. We claim that $x_{k-s+d+r}$ is irregular.

Note that every 1-entry in the interval $x_{k-s}x_{k-s+1}\dots x_{k-s+d}$ sees the three 0-entries x_{k-s}, x_k, x_{k-s+d}. Hence $x_{k-s+d-1} = 1$ and $k-s+d+r < n-d$. Moreover, all further entries seen by $x_{k-s+d-1}$ satisfy $x_{k-s+d+1} = x_{k-s+d+2} = \dots = x_{k-s+2d-1} = 1$. Furthermore, since x_{k+d} sees three 0 entries, $x_{k-s+2d+1} = \dots = x_{k+2d} = 1$. By the definition of s, $x_{k+2d+1} = \dots = x_{k+2d+r-s-1} = 1$. So, indeed, $x_{k-s+d+r}$ sees only the two 0-entries x_{k-s+d} and x_{k-s+2d} and is irregular. $\Box$

We are now prepared to prove Theorem 5.

Proof of Theorem 5:
Let $X^* = X'(\mathbf{u}^*)$ be as in (2). For contradiction, we assume that $X = (x_0, x_1, \dots, x_{n-1})$ is a cyclic binary sequence in $\mathcal{X}$ of smallest order n having minimum density $\mu(X) = g(d, 2d-3)$, and that X is not the concatenation of copies of X^*. Clearly, $\mu(X) \leq \mu(X^*) = \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Since a 1-entry of X must see at least $2d-3$ other 1-entries, we get for $n \leq 2d$ that $\mu(X) = \frac{n_1}{n} \geq \frac{2d-2}{n} \geq 1 - \frac{1}{d} > \mu(X^*)$, a contradiction. So we may assume that $n \geq 2d+1$.

If X contains no pair of overlapping chain ends, then, by Lemma 9 (i), every interval I of length $2d+1$ of X contains an irregular entry. Since every irregular entry contributes to $2d+1$ such intervals, we get by double-counting

$$n \leq (2d+1)n^+, \tag{3}$$

thus, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} > \mu(X^*)$ which is a contradiction.

Hence we may assume that X contains a pair of overlapping ends of chains.

First we assume that X contains more than one such pair. By cyclicity, we may assume that $x_{n-1}x_0$ and $x_{k-1}x_k$ are pairs of overlapping chain ends of X. Let

$$X' = x_0x_1\dots x_{k-1}$$

and

$$X'' = x_kx_{k+1}x_{k+2}\dots x_{n-1}.$$

By Lemma 8 (iii), X' and X'', considered as cyclic sequences, are both in $\mathcal{X}$, because each entry sees the same entries as in X. Since X has minimum density $\mu(X)$ and $\mu(X)$ is a weighted average of the densities $\mu(X')$ and $\mu(X'')$, we obtain $\mu(X') = \mu(X'') = \mu(X)$. Since X' and X'' have smaller lengths than X, by our initial assumption, each of X' and X'' are the concatenation of copies of X^*. Hence X is the concatenation of copies of X^* which is a contradiction.

Therefore, X has exactly one pair of overlapping chain ends, say $x_{n-1}x_0$. Let $\mathcal{J}$ be the set of intervals of length $2d+1$ of X. Let $\mathcal{J}_0 \subseteq \mathcal{J}$ denote the set of those intervals containing a regular chain end and let $\mathcal{J}_2 \subseteq \mathcal{J}$ denote the set of those intervals containing an irregular chain end. By Lemma 9, each interval in $\mathcal{J}_2 \setminus \mathcal{J}_0$ contains at least two irregular entries, while only the intervals in $\mathcal{J}_0 \setminus \mathcal{J}_2$ can contain no irregular entry. If X contains more than one chain, then X contains two different irregular chain ends, hence $|\mathcal{J}_2| \geq 2d+2$ while $|\mathcal{J}_0| \leq 2d+2$. Double-counting the incidences interval/irregular entry we obtain

$$n \leq n + |\mathcal{J}_2| - |\mathcal{J}_0| = n + |\mathcal{J}_2 \setminus \mathcal{J}_0| - |\mathcal{J}_0 \setminus \mathcal{J}_2| \leq (2d+1)n^+,$$

as in (3), which again contradicts $\mu(X) \leq \mu(X^*)$.

So X has a single chain both ends of which overlap. By Lemma 10, X contains at least $d-3$ irregular entries. Hence, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{d-3}{2dn}$. Since $\mu(X) \leq \mu(X^*) = \frac{(2d-3)^2-1}{2(d(2d-3)-1)}$, this implies that

$$n \geq d(2d-3) - 1,$$

i.e. the length of X is at least the length of X^*. By Lemma 8 (iv), each of the $n-(2d+2)$ intervals of length $2d+1$ in $\mathcal{J} \setminus \mathcal{J}_0$ contains at least one irregular entry. Hence $n^+ \geq \frac{n-(2d+2)}{2d+1}$ and, by Lemma 7,

$$\begin{aligned} \mu(X) &\geq \frac{2d-3}{2d} + \frac{n-(2d+2)}{2d(2d+1)n} = \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} - \frac{2d+2}{2d(2d+1)n} \\ &\geq \frac{(2d-3)^2-1}{2(d(2d-3)-1)} = \mu(X^*). \end{aligned}$$

Since $\mu(X) \leq \mu(X^*)$, we obtain $\mu(X) = \mu(X^*)$. Therefore, $n = d(2d-3)-1$, each irregular entry sees exactly $(2d-2)$ 1-entries, and each of the $2d+2$ intervals in $\mathcal{J}_0$ contains no irregular entry while all intervals in $\mathcal{J} \setminus \mathcal{J}_0$ contain exactly one irregular entry. Hence the irregular entries must be exactly $x_{2d+1}, x_{4d+2}, \ldots, x_{(2d+1)(d-3)}$.

So the irregular entries of X and X^*, with the notation of (2), are located at the same positions and, by Lemma 8 (iii), the intervals $x_{n-2d+1} \ldots x_{n-1} x_0 \ldots x_{2d-2}$ of X and $x'_{n-2d+1} \ldots x'_{n-1} x'_0 \ldots x'_{2d-2}$ of X^* are equal.

We assume that for some $i \geq 2d-2$, the intervals $x_{i-2d+1} \ldots x_i$ of X and $x'_{i-2d+1} \ldots x'_i$ of X^* are equal. Now we show that $x_{i+1} = x'_{i+1}$. Indeed, since $x_{i-d+1} = x'_{i-d+1}$ has the same regularity status within X and X^* and sees the same entries in X and X^*, respectively, except possibly at position $i+1$, it follows that $x_{i+1} = x'_{i+1}$. Therefore, $X = X^*$ contradicting the assumption that X is a counterexample. This completes the proof. $\square$

If we define the quantity $\tilde{\delta}(X)$ for a cyclic binary sequence $X = (x_0, x_1, \ldots, x_{n-1})$ as

$$\tilde{\delta}(X) = \min \left\{ \sum_{j=1}^{d} (x_{i+j} + x_{i-j}) \mid 0 \leq i \leq n-1 \right\}$$

and $\tilde{g}(d,k)$ for $d,k \in \mathbb{N}$ with $k \leq 2d$ as the infimum density of a cyclic binary sequence X with $\tilde{\delta}(X) \geq k$, then $g(d,k) \leq \tilde{g}(d,k)$. A simple double-counting implies $\tilde{g}(d,k) \leq \frac{k}{2d}$.

The example described after (1) implies $g(d,k) = \tilde{g}(d,k)$ for $k \geq d+1$ with k even. Furthermore, the comment after Conjecture 4 concerning $k = 2d-1$ and Lemma 6 (i) imply $g(d, 2d-1) = \tilde{g}(d, 2d-1)$ and $g(d, 2d-3) = \tilde{g}(d, 2d-3)$, respectively. Finally, it is easy to check that $\tilde{\delta}(X(\mathbf{u})) \geq k$ for every shifted sequence $X(\mathbf{u})$ for every $\mathbf{u} \in U_k^d$ which does not contain two consecutive 0-entries.

Therefore, Conjecture 4 would - if true - imply that $g(d,k) = \tilde{g}(d,k)$ for all $d+1 \leq k \leq 2d$.

References

[1] J.-C. Bermond and C. Peyrat, Induced subgraphs of the power of a cycle, *SIAM J. Discr. Math.* **2** (1989), 452–455.

[2] B. Bollobás and G. Brightwell, Long cycles in graphs with no subgraphs of minimal degree 3, *Discrete Math.* **75** (1989), 47-53.

[3] P. Erdős, R.J. Faudree, A. Gyárfás, and R.H. Schelp, Cycles in graphs without proper subgraphs of minimum degree 3, *Ars Comb.* **25 B** (1988), 195-201.

[4] P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, Subgraphs of minimal degree k, *Discrete Math.* **85** (1990), 53-58.

[5] P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, Edge conditions for the existence of minimal degree subgraphs, Alavi, Yousef (ed.) et al., Graph theory, combinatorics, and applications, Vol. 1. Proceedings of the sixth quadrennial international conference on the theory and applications of graphs held at Western Michigan University, Kalamazoo, Michigan, May 30-June 3, 1988. New York: John Wiley & Sons Ltd. Wiley-Interscience Publication. 419-434 (1991). ISBN 0-471-60917-X

[6] A. Kézdy, Problem at the problem session of the SIAM meeting in San Francisco, P. Winkler, ed., June 1988.

[7] A. Kézdy, Studies in Connectivity, PhD thesis, University of Illinois at Urbana-Champaign, 1991.

[8] A. Kézdy and M. Markert, Fragile graphs, manuscript 1987.

[9] R.M. Karp, A characterization of the minimum cycle mean in a digraph, *Discrete Math.* **23** (1978), 309-311.